How quaternion algebras shape the structure of square power classes over biquadratic extensions

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In collaboration with...

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The next 25 minutes of your life

Here's what we'll be doing

- Introduce a Galois module of interest
- Review what is known about it
- Reinterpret module-theoretic info arithmetically
- Compute some examples

Motivation and Background

Big picture goal

Problem under consideration

If K/F is a biquadratic extension and $\mathrm{char}(F) \neq 2$, decompose $K^\times/K^{\times 2}$ as module over $\mathbb{F}_2[\mathrm{Gal}(K/F)]$.

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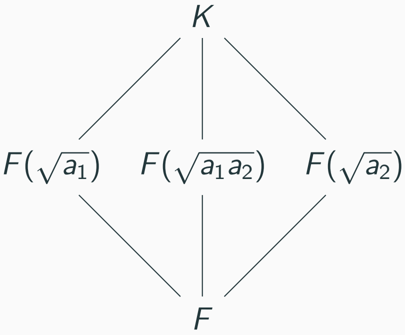
(Spoiler alert: this module has been decomposed, and its "special" for any choice of K/F)

Notation

$K = F(\sqrt{a_1}, \sqrt{a_2})$

$\sigma_i(\sqrt{a_j}) = (-1)^{\delta_{ij}}\sqrt{a_j}$

$G = \mathrm{Gal}(K/F) \simeq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$



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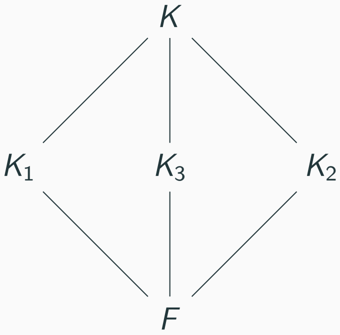

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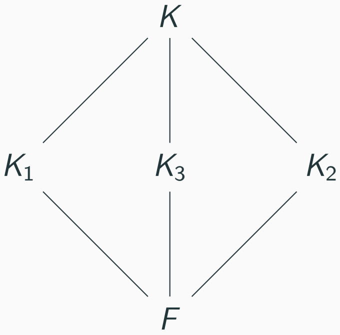

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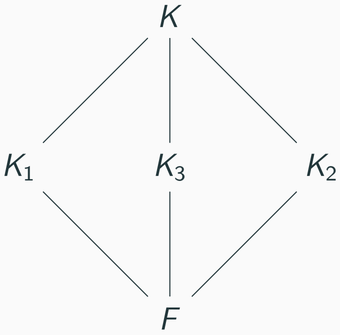
$H_i = \mathrm{Gal}(G/K_i)$



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Key operators: $1 + \sigma_1$ and $1 + \sigma_2$

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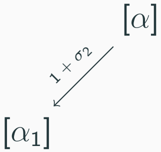

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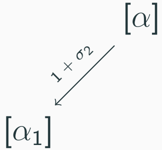

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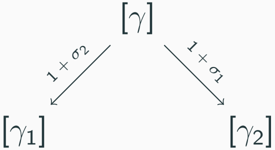

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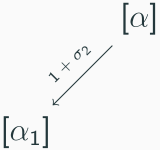

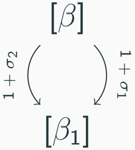

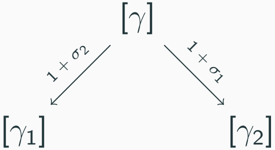

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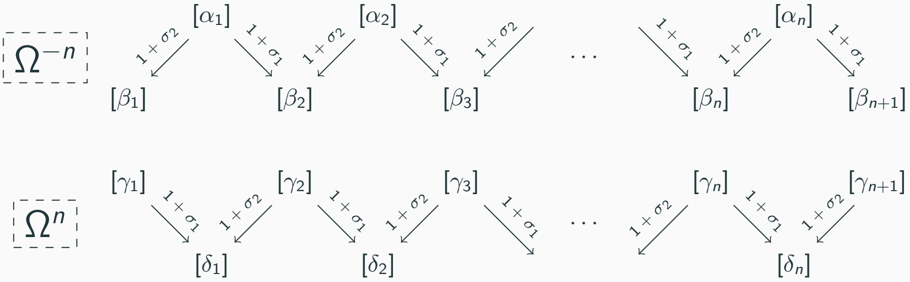
A sample of $\mathbb{F}_2[\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}]$-indecomposables

For $n > 1$, there are 2 indecomposables of dimension $2n+1$



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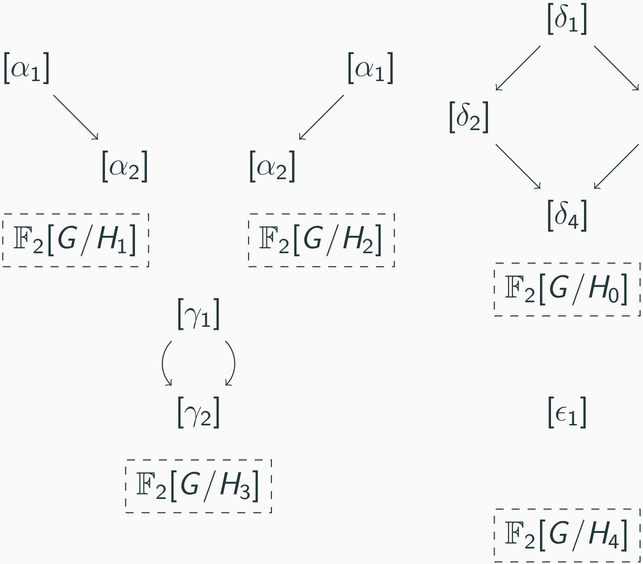

Our module decomposition

Theorem [Chemotti, Mináč, S-, Swallow]

Suppose char$(K) \neq 2$ and $\mathrm{Gal}(K/F) \simeq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. Then

$$K^\times/K^{\times 2} \simeq O_1 \oplus O_2 \oplus Q_0 \oplus Q_1 \oplus Q_2 \oplus Q_3 \oplus Q_4 \oplus X,$$

where

- for each $i \in \{1, 2\}$, the summand O_i is a direct sum of modules isomorphic to Ω^i; and
- for each $i \in \{0, 1, 2, 3, 4\}$, the summand Q_i is a direct sum of modules isomorphic to $\mathbb{F}_2[G/H_i]$; and
- X is isomorphic to one of the following: $\{0\}, \mathbb{F}_2, \mathbb{F}_2 \oplus \mathbb{F}_2, \Omega^{-1}, \Omega^{-2}$, or $\Omega^{-1} \oplus \Omega^{-1}$.

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Motivation and Background

How the decomposition works

Basic strategy

Lemma (Exclusion lemma)

If $U, V \subseteq W$ are $\mathbb{F}_2[G]$-modules, then

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How do we build Y?

Guiding principle

If $[f] \in [F^\times]$ is in the image of a norm map in $K^\times/K^{\times 2}$, make sure it's in the image of that norm map in Y.

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Introducing the norms

$$\begin{array}{ccc} & [k] & \\ \swarrow & & \searrow \\ [k]^{1+\sigma_2} & & [k]^{1+\sigma_1} \\ \searrow & & \swarrow \\ & [f] & \end{array}$$

$$\mathscr{A} = \{[f] : \exists [k] \ni \dots\}$$

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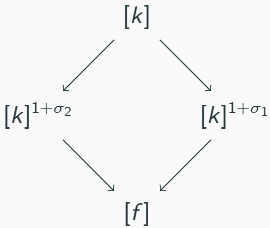

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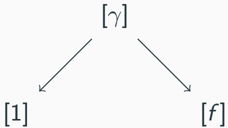
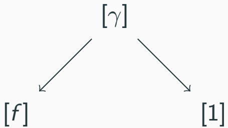


$$\mathscr{B} = \{[f] : \exists [\gamma] \ni \dots\}$$



$$\mathscr{C} = \{[f] : \exists [\gamma] \ni \dots\}$$

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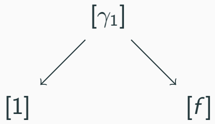

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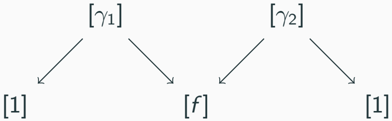

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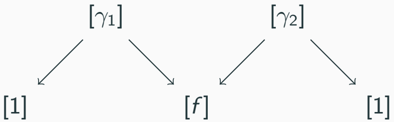

$$\mathscr{V} = \{[f] : \exists [\gamma_1], [\gamma_2] \ni \dots\}$$

To be greedy, we want $\mathscr{V}$ more than $\mathscr{B}$ or $\mathscr{C}$

One final issue

What about $(\mathscr{B} + \mathscr{C}) \cap \mathscr{D}$?

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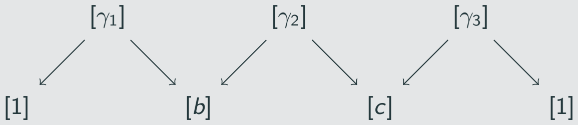
Lemma [Tracking norm interactions]

$[b][c] \in (\mathscr{B} + \mathscr{C}) \cap \mathscr{D}$ if and only if there is a solution to



Define $\mathscr{W} = \{([b],[c]) : \exists [\gamma_1], [\gamma_2], [\gamma_3] \ni \ldots\}$.

Building the unexceptional piece

Proposition

There exists a submodule Y whose fixed part is $[F^\times]$, and which is a direct sum of modules isomorphic to

- $\mathbb{F}_2[G/H_i]$ for $i \in \{0, 1, 2, 3, 4\}$
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Move through subspaces in order $(\mathscr{A}, \mathscr{V}, \mathscr{W}, \mathscr{B}, \mathscr{C}, \mathscr{D}, [F^\times])$

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Reinterpreting the construction of Y

Arithmetic interpretation for solvability

Original argument views Y in terms of solvability of diagrams, but gives no indication of how we determine solvability

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Theorem [Diagram solvability and $\mathrm{Br}(F)$]

Let $\mathcal{S} = \langle (a_1, a_1), (a_1, a_2), (a_2, a_2) \rangle \subseteq \mathrm{Br}(F)$. For $f, g \in F^\times$, we have $(a_1, f)(a_2, g) \in \mathcal{S}$ iff there exists $\gamma \in K^\times$ with

$$[g] \xleftarrow{1+\sigma_2} [\gamma] \xrightarrow{1+\sigma_1} [f]$$

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Sketch of proof: solvability of Galois embedding problems

Thinking rationally

Great news: if $F = \mathbb{Q}$, then local-global principle makes computing elements of $\mathrm{Br}(\mathbb{Q})$ nicely explicit:
$(a, b) = (c, d) \in \mathrm{Br}(\mathbb{Q})$ iff for all $v \in \{2, 3, 5, 7, \cdots, \infty\}$ we have $(a, b)_v = (c, d)_v$

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- if $p = \infty$ and $a, b \in \mathbb{Z}$ then

$$(a, b)_\infty = -1 \text{ if } a, b < 0, \qquad (a, b)_\infty = 1 \text{ else}$$

- if p odd prime then for $\gcd(a, p) = \gcd(b, p) = 1$ we get

$$(a, b)_p = 1, \qquad (a, p)_p = \left(\frac{a}{p}\right), \qquad (p, p)_p = \left(\frac{-1}{p}\right)$$

- if $p = 2$ and $a, b \in 2\mathbb{Z} + 1$ then

$$(a, b)_2 = (-1)^{\frac{a-1}{2} \cdot \frac{b-1}{2}}, \quad (a, 2)_p = (-1)^{\frac{a^2-1}{8}}, \quad (2, 2)_2 = 1$$

Application: hunting for summands

$$\mathscr{V} = \left\{ [f] : \exists [\gamma_1], [\gamma_2] \text{ with } \begin{array}{ccccccc} & & [\gamma_1] & & [\gamma_2] & & \\ & \swarrow & & \searrow \; \swarrow & & \searrow & \\ [1] & & & [f] & & & [1] \end{array} \right\}$$

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Corollary

Ω^1 summands of $K^\times/K^{\times 2}$ exist if there exists f so that $(a_1, f), (a_2, f) \in \mathcal{S} \setminus \{0\}$.

Finding Ω^1 summands in the wild

Let $K/F = \mathbb{Q}(\sqrt{7}, \sqrt{-5})/\mathbb{Q}$

$$\mathcal{S} = \langle (7,7), (7,-5), (-5,-5) \rangle$$

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Goal: show $K^\times / K^{\times 2}$ has Ω^1 summands

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Strategy: find prime p with $(-5, -p) = (-5, -5)$ and $(7, -p) = (7, 7)$

Finding our prime, part I: $(-5,-5)=(-5,-p)$

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So we want $p \equiv 1 \pmod 4$ and $p \equiv 1, 4 \pmod 5$

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$\rightsquigarrow$ lots of Ω^1 summands in this module

What about Ω^2 summands?

Ω^2 summands occurs for solutions to

$$\begin{array}{ccccccccccc} & & [\gamma_1] & & & & [\gamma_2] & & & & [\gamma_3] & & \\ & \swarrow & & \searrow & & \swarrow & & \searrow & & \swarrow & & \searrow & \\ [1] & & & & [f] & & & & [g] & & & & [1] \end{array}$$

AND we must have $[f], [g] \notin \mathscr{V}$

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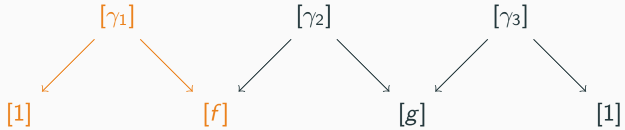

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So we need $(a_1, f), (a_2, g) \in \mathcal{S}$ and $(a_2, f)(a_1, g) \in \mathcal{S}$ but $(a_2, f), (a_1, g) \notin \mathcal{S}$

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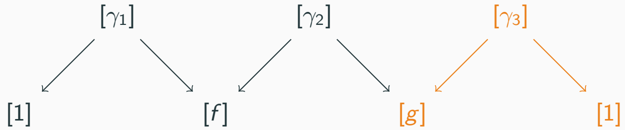

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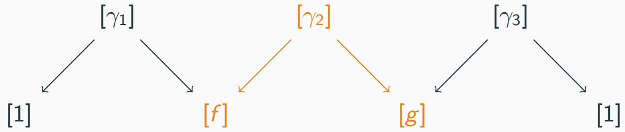

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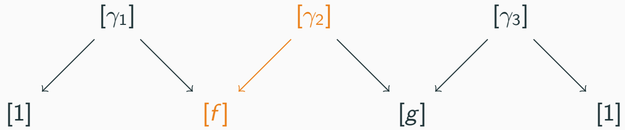

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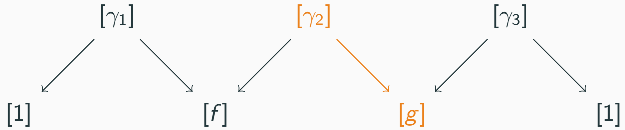

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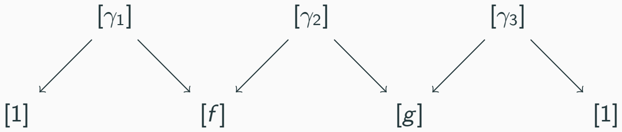

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Corollary

Ω^2 summands of $K^\times / K^{\times 2}$ exist if there exist f, g so that $(a_1, f), (a_2, g) \in \mathcal{S}$ and $(a_1, g) = (a_2, f) \notin \mathcal{S}$.

Finding Ω^2 summands in the wild

Let $K/F = \mathbb{Q}(\sqrt{33}, \sqrt{35})/\mathbb{Q}$

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Strategy: find primes p, q with $(33, 3pq) = (33, 33)$ and $(35, 7pq) = (1, 1)$ and $(33, 7pq) = (35, 3pq) \notin \mathcal{S}$

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$\rightsquigarrow$ Choose p so $p \not\equiv \square \pmod 3$, $p \not\equiv \square \pmod 4$, $p \not\equiv \square \pmod 5$, $p \equiv \square \pmod 7$, and $p \not\equiv \square \pmod{11}$

$\rightsquigarrow$ Choose q so $q \equiv \square \pmod 3$, $q \equiv \square \pmod 4$, $q \equiv \square \pmod 5$, $q \equiv \square \pmod 7$, and $q \equiv \square \pmod{11}$

Lather, rinse, repeat

This same strategy provides methods for realizing other "unexceptional" summand types over well-chosen rational biquadratic extensions

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The structure of the X summand also has new interpretation in this lens (but less exciting since it was originally interpretable in terms of Galois embeddings)

Thanks!